Directed Network Design, Flows, and Matroids

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In the first three chapters, we prove relaxations and special cases of Woodall's conjecture. In a digraph, a dicut is a cut where all the arcs cross in one direction. A dijoin is a subset of arcs that intersects each dicut. It has been proven that the minimum size of a dicut, say τ, is equal to the maximum fractional packing of dijoins. Woodall conjectured in 1976 that the maximum integral packing is also equal to k. In other words, it is conjectured that the minimum size of a dicut is equal to the maximum number of disjoint dijoins. First, we show that Woodall's conjecture is true up to a constant factor. More precisely, there exist $\tau/6$ disjoint dijoins. Motivated by the attempt to improve the approximation factor for packing dijoins, we study rooted cut-balanced orientations. We show that rooted cut-balanced orientations are equivalent to cut-balanced orientations. While the weighted version of Woodall's conjecture is not true for general digraphs, we show that it is true for digraphs whose underlying undirected graph is chordal.

In the fourth chapter, we consider relaxing the nonnegativity of the packing coefficients in Woodall's conjecture and show that there exists an integral packing of τ dijoins.

Motivated by the use of nowhere-zero flows in our first chapter, in the fifth chapter, we study the min-cost version of nowhere-zero flows and cut-balanced orientations. Given a bidirected graph with asymmetric costs on two orientations of each edge, we study the min-cost nowhere-zero k-flow and min-cost k-cut-balanced orientation problems. We show strong inapproximability results for both problems and design bicriteria approximation algorithms.

In the sixth chapter, we study congruency-constrained matroid optimization problems. Consider a matroid where all elements are labeled with an integer. We are interested in finding a base where the sum of the labels is congruent to g modulo m. We show that this problem can be solved in deterministic FPT time parameterized by m, when m is either the product of two primes or a prime power. The algorithm can be generalized to all moduli and to all abelian groups, if a classic additive combinatorics conjecture by Schrijver and Seymour holds true.

In the seventh chapter, we study the smallest support size of integer solutions to linear equations. We show that the smallest support size of solutions to $Ax = b$, $x \in \mathbb{Z}^n$ where $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^n$ is $O\big(m \log(\sqrt{m}\,\|A\|_\infty) \big/ \log\log(\sqrt{m}\,\|A\|_\infty)\big)$. This bound is asymptotically tight, and matches the bound given in Aliev, Averkov, De Loera and Oertel, 2022, while our proof is simpler, relying only on linear algebra.

In the eighth chapter, we consider special cases of a conjecture by Morell and Skutella on unsplittable flows with arc-wise upper and lower bounds that can be framed as follows: Given positive integers m, n, a fractional assignment $x \in [0,1]^{m\times n}$ so that $\sum_{i=1}^{m} x_{ij} = 1\ \forall j$ and weights $d \in \mathbb{R}^n_{>0}$, we show that there exists an assignment $y \in \{0,1\}^{m\times n}$ so that $\sum_{i=1}^{m} x_{ij} = 1\ \forall j$ and $\left|\sum_{j\in[t]} d_j(x_{ij} - y_{ij})\right| \leq \max_{j\in[n]} d_j\ \forall i \in [m], t \in [n]$. This provides an alternate proof of a 3-approximation for maximum flow-time scheduling for identical machines and generalizes to the setting when machines have closing times.

Next, we describe ongoing work. We propose to study the weighted matroid equitability conjecture: given a matroid whose ground set E decomposes into k bases and weights on E, there exists a decomposition into k bases $B_1, \ldots, B_k$ such that $|w(B_i) - w(E)/k| \leq 1/2 \cdot \max_{j\in E} w_j$. This is a relaxation of Gabow's conjecture and the unweighted version where $w \in \{0,1\}^E$ was recently proved.

We also propose to study approximation algorithms for network design problems. We work on improving the integrality gap of LP relaxations of the weighted tree augmentation problem. We propose and study the weighted directed tree augmentation problem: given an arbitrarily oriented tree, find a subset of directed non-tree arcs (links) with minimum costs to cover all tree arcs, where a directed link covers a tree arc if they go in the same direction in the fundamental circuit formed by adding the link to the tree.